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New Renormalization Procedure for Eliminating Redundant Operators

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A generalized exact renormalization group equation is obtained by using a new rencrmalization procedure. This equation does not contain redundant operators and therefore enables one to avoid using an uncertain procedure for their exclusion.

KEY WORDS: Renormalization group; redundant operators; phase transitions.

The renormalization group (RG) is now the most effective tool for the investigation of critical phenomena. It allows one to develop various approaches which use different kinds of perturbation theory, such as the ε-expansion,[1] the expansion in the inverse component number of a vector order parameter,[2,3] the expansion in coupling constants in three-dimensional space,[4,5] and others. Critical exponents with a very high degree of accuracy were obtained using these approaches.[6-8] Besides the approaches mentioned, some approximation schemes which do not use a perturbation theory have also been developed.[9-12] Most of them explore the exact RG equation obtained by Wilson[13] (see also refs. 14, 15). These approaches as well as ones based on a perturbation theory have led to very good values of critical exponents.[16] Another important advantage of using exact RG equations is that they may be useful for the foundation of perturbation methods used in RG theory.

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In this paper we obtain a generalized exact RG equation. This equation contains an arbitrary function which can be used to eliminate redundant operators, to introduce a small parameter for the equation, to simplify the initial Ginzburg–Landau–Wilson functional, etc. The equation obtained can serve as a basis for constructing new approximation schemes in the theory of critical phenomena. One of these schemes, using the exponent η as a small parameter, was developed in ref. 17.

Let us consider the Ginzburg–Landau–Wilson functional for a translationally invariant isotropic system,

$$H_I[\vec{\phi}] = \sum_{k=0}^{\infty} 2^{1-2k} \int_{q_1, q_{\bar{1}}; \ldots; q_k, q_{\bar{k}}} g_k(\mathbf{q}_1, \mathbf{q}_{\bar{1}}; \ldots; \mathbf{q}_k, \mathbf{q}_{\bar{k}})(2\pi)^d \times \delta\left(\sum_{i=1}^{k} (\mathbf{q}_i + \mathbf{q}_{\bar{i}})\right) \prod_{i=1}^{k} (\vec{\phi}(\mathbf{q}_i) \cdot \vec{\phi}(\mathbf{q}_{\bar{i}})) \tag{1}$$

where $\vec{\phi}$ is an n-component vector, the vertices $g_k(\mathbf{q}_1, \mathbf{q}_{\bar{1}}; \ldots; \mathbf{q}_k, \mathbf{q}_{\bar{k}})$ are invariant with respect to permutations of any pairs of momenta $\mathbf{q}_i$, $\mathbf{q}_{\bar{i}}$ and $\mathbf{q}_j$, $\mathbf{q}_{\bar{j}}$ with each other and among themselves,

$$\delta(\mathbf{q}) = (2\pi)^{-d} \delta_{q,0} V, \qquad \int_q = V^{-1} \sum_q = \int \frac{d^d q}{(2\pi)^d}$$

V is the system volume.

To derive an RG equation, one has, first of all, to perform an integration over short-wave modes in a partition function. For this purpose, we add to the functional (1) the term

$$H_0[\vec{\phi}] = \frac{1}{2} \int_q G_0^{-1}(q, \Lambda)\, |\vec{\phi}(\mathbf{q})|^2 \tag{2}$$

where the propagator G_0 is defined as

$$G_0(q, \Lambda) = q^{-2} S(q^2/\Lambda^2) \tag{3}$$

The function $S(x)$ provides a momentum cutoff on a momentum Λ. It is monotonic with $S(x=0)=1$ and $\lim_{x\to\infty} S(x)\, x^m = 0$, for any m. In particular, the choice of $S(x) = \Theta(1-x)$, where Θ is the step function, provides a sharp cutoff.

Now let us note that the partition function of a system with the functional $H = H_0 + H_I$ can be written in the form

$$Z = \int D\vec{\phi} \exp(-H[\vec{\phi}]) = Z_0 \langle \exp(-H_I[\vec{\phi}]) \rangle_{0,\Lambda} \equiv Z_0 \langle w[\vec{\phi}] \rangle_{0,\Lambda} \tag{4}$$

where

$$Z_0 = \int D\vec{\phi} \exp(-H_0[\vec{\phi}])$$

and averaging $\langle \cdots \rangle_{0,\Lambda}$ is performed with the Gaussian functional at a given value of Λ.

The following considerations are based on the fact that the averaging over a Gaussian field $\vec{\phi}$ can be replaced by two independent averages over Gaussian fields $\vec{\phi}_1$ and $\vec{\phi}_2$, providing $\vec{\phi} = \vec{\phi}_1 + \vec{\phi}_2$ and the sum of the correlators $G_{01}(q, \Lambda_1) = \langle |\vec{\phi}_1(\mathbf{q})|^2 \rangle_{0,\Lambda_1}$ and $G_{02}(q, \Lambda_2) = \langle |\vec{\phi}_2(\mathbf{q})|^2 \rangle_{0,\Lambda_2}$ is equal to the correlator of the initial field

$$G_0(q, \Lambda) = \langle |\vec{\phi}(\mathbf{q})|^2 \rangle_{0,\Lambda} = G_{01}(q, \Lambda_1) + G_{02}(q, \Lambda_2)$$

Let us choose $\vec{\phi}_1$ so that $G_{01}(q, \Lambda_1) = G_0(q, (1-\xi)\Lambda)$, where $\xi \ll 1$. Then the value G_{02} is small, of the order of ξ:

$$G_{02}(q, \Lambda_2) = G_0(q, \Lambda) - G_{01}(q, \Lambda_1) \simeq \xi \cdot \Lambda \frac{\partial G_0(q, \Lambda)}{\partial \Lambda} \equiv 2\xi h(q)$$

$$h(q) = q^{-2} \Lambda^2 \frac{dS(q^2/\Lambda^2)}{d\Lambda^2} \tag{5}$$

In this case the integration over the field ϕ_2 in Eq. (4) can be performed easily:

$$Z = Z_0 \langle w[\vec{\phi}] \rangle_{0,\Lambda} = Z_0 \langle (1 + \xi \hat{L}_A) \, w[\vec{\phi}] \rangle_{0,\Lambda(1-\xi)} \tag{6}$$

The operator $\hat{L}_A$ in Eq. (6) is defined by the relationship

$$\hat{L}_A = \int_q h(q) \frac{\delta^2}{\delta\vec{\phi}(\mathbf{q}) \cdot \delta\vec{\phi}(-\mathbf{q})} \tag{7}$$

The RG transformation will be completed by changing the momentum scale in order to restore the initial value of the parameter Λ: $\mathbf{q} = \mathbf{q}'(1+\xi)$. One should also transform the field $\phi(\mathbf{q})$:

$$\phi(\mathbf{q}) = [1 + \xi \varepsilon_\phi(\mathbf{q}')] \, \vec{\phi}'(\mathbf{q}') = [1 + \xi(\varepsilon_\phi(\mathbf{q}) + \mathbf{q}\nabla_q)] \, \phi'(\mathbf{q}) \tag{8}$$

If one chooses $\varepsilon_\phi(\mathbf{q}) = (d+2)/2$, then the functional H_0 will be restored. This results in the following equation:

$$w'[\vec{\phi}] = [1 + \xi(\hat{L}_A + \hat{L}_B + \hat{L}_V)] \, w[\vec{\phi}] \tag{9}$$

where

$$\hat{L}_B = \left(\frac{d+2}{2} + \mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right) \vec{\phi}(\mathbf{q}) \cdot \frac{\delta}{\delta \vec{\phi}(\mathbf{q})} \tag{10}$$

and the operator

$$L_V = dV \frac{\partial}{\partial V}$$

appears due to the transformation of the volume of the system.

From Eq. (9) we obtain

$$\begin{aligned} \dot{H}_I[\phi] = {} & dV \frac{\partial H_I}{\partial V} + \int_q \left[\frac{d+2}{2} \vec{\phi}(\mathbf{q}) + \mathbf{q} \cdot \frac{\partial \vec{\phi}(\mathbf{q})}{\partial \mathbf{q}} \right] \cdot \frac{\delta H_I[\vec{\phi}]}{\delta \vec{\phi}(\mathbf{q})} \\ & + \int_q h(\mathbf{q}) \left[\frac{\delta^2 H_I[\vec{\phi}]}{\delta \vec{\phi}(\mathbf{q}) \cdot \delta \vec{\phi}(-\mathbf{q})} - \frac{\delta H_I[\vec{\phi}]}{\delta \vec{\phi}(\mathbf{q})} \cdot \frac{\delta H_I[\vec{\phi}]}{\delta \vec{\phi}(-\mathbf{q})} \right] \end{aligned} \tag{11}$$

Equation (11) is an exact RG equation. It differs slightly from the equation obtained in refs. 12–15. One can obtain all known results of RG theory using this relation. However, this equation is not convenient for practical uses. As with Wilson's equation, it contains redundant operators[15] which should be properly excluded to obtain results having physical meaning. Now we are going to obtain an equation free of this substantial drawback. It is not necessary to restore the initial functional. If one is not restricted to choosing the constant $(d+2)/2$ for the function $\varepsilon_\phi(q)$, then an additional term will appear in Eq. (11). Still this will be an exact RG equation. Let us define $\varepsilon_\phi(q)$ as

$$\varepsilon_\phi(q) = \frac{d+2}{2} - \eta(q)$$

Then instead of Eq. (9) we have

$$\begin{aligned} w'[\vec{\phi}] = \Bigg[1 + \xi \bigg(& \hat{L}_A + \hat{L}'_B + \hat{L}_V + \frac{1}{2} \int_q \eta(q)\, G_0^{-1}(q, \Lambda)\, |\vec{\phi}(\mathbf{q})|^2 \\ & - \frac{1}{V} \int_q \eta(q) \bigg) \Bigg] w[\vec{\phi}] \end{aligned} \tag{12}$$

where the operator $\hat{L}'_B$ is defined as

$$\hat{L}'_B = \int_q \left[\frac{d+2-\eta(q)}{2} \vec{\phi}(\mathbf{q}) + \mathbf{q} \cdot \frac{\partial \vec{\phi}(\mathbf{q})}{\partial \mathbf{q}} \right] \cdot \frac{\delta}{\delta \vec{\phi}(\mathbf{q})} \tag{13}$$

Using Eqs. (12), (4), (7), and (13), we find

$$\dot{H}_I[\vec{\phi}] = Vd\frac{\partial H_I[\vec{\phi}]}{\partial V} + \frac{V}{2}\int_q \eta(q) - \frac{1}{2}\int_q \eta(q)\, G_0^{-1}(q, \Lambda)\, |\vec{\phi}(\mathbf{q})|^2$$

$$+ \int_q \left[\frac{d+2-\eta(q)}{2}\vec{\phi}(\mathbf{q}) + \mathbf{q}\frac{\partial \phi(\mathbf{q})}{\partial \mathbf{q}}\right] \cdot \frac{\delta H_I[\vec{\phi}]}{\delta \vec{\phi}(\mathbf{q})}$$

$$+ \int_q h(q)\left[\frac{\delta^2 H_I[\vec{\phi}]}{\delta\vec{\phi}(\mathbf{q})\cdot\delta\vec{\phi}(-\mathbf{q})} - \frac{\delta H_I[\vec{\phi}]}{\delta\vec{\phi}(-\mathbf{q})}\cdot\frac{\delta H_I[\vec{\phi}]}{\delta\vec{\phi}(\mathbf{q})}\right] \tag{14}$$

Again we have obtained the exact RG equation. This equation is a generalized form of Eq. (11). Equation (14) contains an arbitrary function $\eta(q)$ which allows one to exclude redundant operators from a set of RG eigenoperators. The presence of this function was essentially explored by authors to construct a new small-parameter perturbation theory for phase transitions.[17] When $\eta(q) = 0$, Eq. (14) immediately reduces to Eq. (11). In the rest of this section we show how the function $\eta(q)$ can be used to eliminate the generation of a momentum-dependent part of the vertex $g_2(q)$ is the RG equation. That makes the definition of the RG transformation unique because it forbids changes of the nonlocal part of the initial Hamiltonian. This procedure not only excludes redundant operators, but also defines the value of $\eta(q)$, leaving no free parameters in the theory.

Let us separate terms of zeroth and first order in ϕ^2 in the functional H_I:

$$H_I[\vec{\phi}] = bV + \frac{1}{2}\int_q g_1(q)\, |\vec{\phi}(\mathbf{q})|^2 + H' \tag{15}$$

Then, for a renormalization of the constant b, one obtains the simple equation

$$\dot{b} = db + \frac{1}{2}\int_q \eta(q) + n\int_q h(q)\, g_1(q) \tag{16}$$

The equation for $g_1(q)$ has the form

$$\dot{g}_1(q) = -\eta(q)\, G_0^{-1}(q, \Lambda) + \left[2 - \eta(q) - 2q^2\frac{\partial}{\partial q^2}\right] g_1(q) + Q(q) - 2h(q)\, g_1^2(q) \tag{17}$$

where the function $Q(q)$ is given by

$$Q(q) = \int_p h(p)[ng_2(-\mathbf{p}, \mathbf{p}; -\mathbf{q}, \mathbf{q}) + 2g_2(-\mathbf{p}, \mathbf{q}; \mathbf{p}, -\mathbf{q})] \tag{18}$$

Now let us separate the part dependent on momenta from the vertex $g_1(q) = g_{10} + g_1'(q)$. Equations for g_{10} and $g_1'(q)$ can be written in the form

$$\dot{g}_{10} = [2 - \eta(0)]\, g_{10} + Q(0) - 2h(0)\, g_{10} \tag{19}$$

$$\dot{g}_1'(q) = -\eta(q)\, G_0^{-1}(q, \Lambda) - [\eta(q) - \eta(0)]\, g_{10} + Q(q) - Q(0)$$
$$- 2[h(q) - h(0)]\, g_{10} + 2q^2 \frac{\partial g_1'(q)}{\partial q^2} - 2h(q)\, g_1'^2(q) \tag{20}$$

Our objective is to eliminate the generation of the q-dependent part of the vertex g_1 in the RG equation provided that its initial value does not depend on momentum [i.e., $\dot{g}_1'(q) = 0$, $g_1'(q) = 0$]. Using Eq. (20), we can easily find the function $\eta(q)$ which ensures such behavior,

$$\eta(q) = \eta(0) - \frac{D(q) - D(0) + \eta(0)\, G_0^{-1}(q, \Lambda)}{G_0^{-1}(q, \Lambda) - g_{10}} \tag{21}$$

where $D(q) = Q(q) - 2h(q)\, g_{10}$. In addition, if we demand that

$$\eta(0) = \left. \frac{dD(q)}{dq^2} \right|_{q=0} \tag{22}$$

then the expansion of $\eta(q)$ will start from q^4. Now the function $\eta(q)$ depends on all higher vertices and has clear physical meaning: at a stable fixed point (critical behavior) $\eta^*(0)$ is equal to the Fisher exponent.

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